§ Integral Calculus on Surfaces.

We have talked about differential calculus on surfaces.

Now, we move on to integration.

Question: Given a function $f: S \to \mathbb{R}$ on a surface S,

how to define $\int_S f$?

Some properties about integration on $\mathbb{R}^2$:

(1) For any bounded subset $\mathcal{U} \subseteq \mathbb{R}^2$,

$$\int_{\mathcal{U}} 1 = \text{Area}(\mathcal{U})$$

(2) Change of variable formula:

$$\boxed{\int_{\mathcal{U}} f(x,y)\, dx\, dy = \int_{\mathcal{U}'} f(u,v)\, |\text{Jac}\, \phi|\, du\, dv}$$

where $\phi: \mathcal{U}' \to \mathcal{U}$ is the change of coordinate transformation

$$\phi(u,v) = (x(u,v), y(u,v))$$

with Jacobian determinant

$$\text{Jac}\, \phi := \det(d\phi) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

E.g. $|\text{Jac}\, \phi| = r$ for polar coordinates r, θ

(3) $\exists$ various "Fundamental Theorems of Calculus"

Green's Theorem, Stokes' Theorem, Divergence Theorem

$$\boxed{\int_{\Omega} \text{"}d\text{"}\omega = \int_{\partial\Omega} \omega}$$

Def$^{\underline{n}}$: The support of a function $f: S \to \mathbb{R}$ is defined as

$$\mathrm{spt}(f) := \overline{\{x \in S : f(x) \neq 0\}}$$ closure in S

Def$^{\underline{n}}$: Let $\mathbf{X}: \mathcal{U} \xrightarrow{\cong} V \subseteq S$ be a parametrization of S

and $f: S \to \mathbb{R}$ be a function s.t. $\mathrm{spt}(f) \subseteq V$

Define

$$(*) \ldots \quad \int_S f := \int_{\mathcal{U}} f \circ \mathbf{X} \left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\| du\,dv$$

Remark: The def$^{\underline{n}}$ is independent of the choice of $\mathbf{X}$, i.e.

if $\mathbf{X}': \mathcal{U}' \xrightarrow{\cong} V$ is another parametrization, then

$$\int_{\mathcal{U}} f \circ \mathbf{X} \left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\| du\,dv = \int_{\mathcal{U}'} f \circ \mathbf{X}' \left\| \frac{\partial \mathbf{X}'}{\partial u'} \times \frac{\partial \mathbf{X}'}{\partial v'} \right\| du'dv'$$

$\because$ change of variable formula (Ex: Prove this!)

Note: If the function f is not supported on a single coordinate neighborhood, i.e. $\operatorname{spt}(f) \not\subseteq V$, one can use a "partition of unity" to decompose into a (finite) sum

$$f = \sum_{\alpha} f_\alpha \qquad \text{s.t.} \quad \operatorname{spt}(f_\alpha) \subseteq V_\alpha$$

s.t. each f_α is contained in a single coord. nbd. V_α. Then,

$$\int_S f := \sum_\alpha \int_S f_\alpha$$

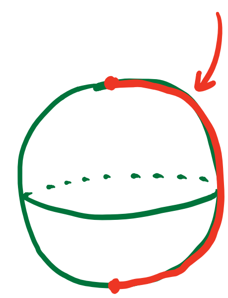
In practice, if $\mathbb{X} : \mathcal{U} \to S$ is a parametrization which covers almost all of S (except a set of "measure zero") then $(*)$ is still applicable.

Example: (Area of the sphere)

The parametrization $\mathbb{X} : (0, 2\pi) \times (0, \pi) \longrightarrow \mathbb{S}^2(r)$

$$\mathbb{X}(\theta, \varphi) := (r \sin\varphi \cos\theta \,, r \sin\varphi \sin\theta \,, r\cos\varphi)$$

covers almost the whole sphere $\mathbb{S}^2(r)$ except for a latitude



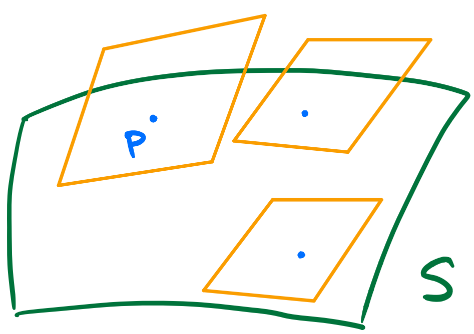
$$\begin{aligned}
\operatorname{Area}(\mathbb{S}^2(r)) &= \int_{\mathbb{S}^2(r)} 1 \\
&= \int_0^\pi \int_0^{2\pi} 1 \cdot \underbrace{r^2 \sin\varphi}_{\left\| \frac{\partial \mathbb{X}}{\partial \theta} \times \frac{\partial \mathbb{X}}{\partial \varphi} \right\|} \, d\theta \, d\varphi \\
&= 4\pi r^2.
\end{aligned}$$

§ First Fundamental Form

Recall that we have defined at each p on a surface S the tangent plane T_pS, which is a 2-dimensional subspace of $\mathbb{R}^3$. By putting all these tangent planes together, we get

tangent bundle of S : $$\boxed{TS := \{(p, v) : p \in S,\ v \in T_pS\}}$$

Note: We can think of TS as a 2-parameter family of 2-dimensional vector spaces (i.e. T_pS) parametrized by points p on S.



"disjoint union"

$$TS = \coprod_{p \in S} T_pS$$

Since each T_pS is a subspace of $\mathbb{R}^3$, it inherits the inner product from $\mathbb{R}^3$ as well. Therefore, we have the following:

Def$^{\underline{n}}$: The first fundamental form (1^{st} f.f.) of a surface at a point $p \in S$ is a positive definite, symmetric bilinear form (i.e. an inner product) defined on T_pS by

$$g_p : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{g_p(u,v) := \langle u, v \rangle_{\mathbb{R}^3}}$$

standard inner product in $\mathbb{R}^3$

Note: TS is then a smooth family of inner product spaces parametrized by S.

We can express the 1^{st} f.f. locally as 2×2 matrices (g_{ij}) using coordinate systems as follows:

Given a parametrization $X(u_1, u_2) : \mathcal{U} \longrightarrow S$,

$$T_pS = \mathrm{Span}\left\{ \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \right\}$$

we can express g_p by a matrix as

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{where} \quad g_{ij} = \left\langle \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} \right\rangle \quad (i,j = 1,2)$$

2×2 symmetric matrix

Therefore, if $u = a \frac{\partial}{\partial u_1} + b \frac{\partial}{\partial u_2}$ where $\frac{\partial}{\partial u_i} = \frac{\partial \mathbf{X}}{\partial u_i}$

$v = c \frac{\partial}{\partial u_1} + d \frac{\partial}{\partial u_2}$

then

$$g(u, v) = (a \quad b) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

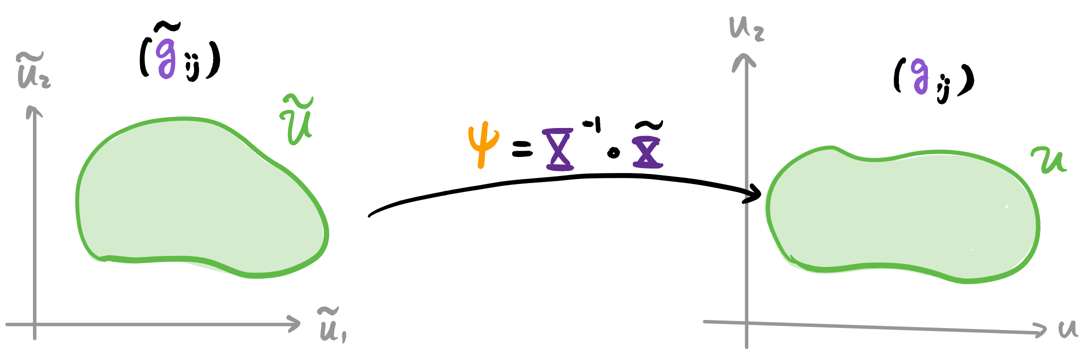
Question: How does the matrix (g_{ij}) transform when we change local coordinates?

Lemma: (Transformation law for (g_{ij}))

Suppose (g_{ij}) and $(\tilde{g}_{ij})$ are the 1^{st} f.f. expressed in local coordinates $\mathbf{X}(u_1, u_2) : \mathcal{U} \to S$ and $\tilde{\mathbf{X}}(\tilde{u}_1, \tilde{u}_2) : \tilde{\mathcal{U}} \to S$ respectively. Then,

$$\boxed{(\tilde{g}_{ij}) = (D\psi)^T (g_{ij}) (D\psi)}$$

where $\psi = \mathbf{X}^{-1} \circ \tilde{\mathbf{X}} : \tilde{\mathcal{U}} \to \mathcal{U}$ is the transition map.



Proof: First of all,

$$(g_{ij}) = \begin{pmatrix} \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \rangle \\ \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2} \rangle \end{pmatrix} \quad 2\times 2$$

$$= \underbrace{\begin{pmatrix} — \frac{\partial X}{\partial u_1} — \\ — \frac{\partial X}{\partial u_2} — \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} | & | \\ \frac{\partial X}{\partial u_1} & \frac{\partial X}{\partial u_2} \\ | & | \end{pmatrix}}_{3\times 2} = (DX)^T (DX)$$

On the other hand,

$$(\tilde{g}_{ij}) = (D\tilde{X})^T (D\tilde{X})$$

$$= \underbrace{(D\psi)^T (DX)^T}_{} \underbrace{(DX)(D\psi)}_{} \qquad (\because \tilde{X} = X \circ \psi)$$

$$= (D\psi)^T (g_{ij}) (D\psi)$$

$\square$

Corollary: $\sqrt{\det(\tilde{g}_{ij})} = \sqrt{\det(g_{ij})}\ |\mathrm{Jac}\,\psi|$

Note: $\sqrt{\det(g_{ij})} = \left\| \frac{\partial X}{\partial u_1} \times \frac{\partial X}{\partial u_2} \right\|$

$$\Rightarrow \int_S f \underset{\text{locally}}{=} \int_{\mathcal{U}} f \underbrace{\sqrt{\det(g_{ij})}\, du_1 du_2}_{dA\,:\ \text{area form.}}$$